DIRECT PRODUCT OF GROUPS

Let N_1 and N_2 be 2 subgroups of G such that

(i) $\forall m_1 \in N_1$, $\forall m_2 \in N_2$ $\quad m_1 \cdot m_2 = m_2 \cdot m_1$

(ii) $\forall g \in G$ we can write $g = m_1 \cdot m_2$ in a UNIQUE way with $m_1 \in N_1$, $m_2 \in N_2$

$\Rightarrow$ The group G is said to be the DIRECT PRODUCT of N_1 and N_2

and we write $G = N_1 \otimes N_2$

$g \sim (m_1, m_2) \qquad g' \sim (m_1', m_2') \qquad gg' = (m_1 \cdot m_1', m_2 \cdot m_2')$

$\Rightarrow$ analogue of the Cartesian product of sets

EXAMPLE

transformations that leave the length of a 3-vector invariant

$$O(3) = \underbrace{SO(3)}_{\text{3-dim rotations}} \otimes \underbrace{\{I, P\}}_{C_2}$$

identity and parity ($\underline{r} \to -\underline{r}$)

If $G = N_1 \otimes N_2 \Rightarrow N_1$ and N_2 are invariant subgroups

$\boxed{\text{GROUP REPRESENTATIONS}}$

group theory in physics $\rightarrow$ symmetry transformations on physical systems

classical physics $\rightarrow$ how solutions of integral or differential equations transform under the symmetry $\rightarrow$ usually form a vector space

quantum physics $\rightarrow$ a linear vector space is adopted as a basic space to describe the quantum system (HILBERT SPACE)

The interest of group theory centers on the way in which the SYMMETRY is realized on the physical system

SYMMETRY OF A PHYSICAL SYSTEM	$\leftrightarrow$	GROUP
IMPLEMENTATION OF THE SYMMETRY	$\leftrightarrow$	GROUP REPRESENTATION

DEF : HOMOMORPHISM

A homomorphism is a mapping $G \to G'$ (not necessarily one-to-one) which preserves the product, that is if $g_1 g_2 = g_3 \Rightarrow g_1' g_2' = g_3'$

DEF : GROUP REPRESENTATION

Let us consider a group of operators U on a linear vector space V. We say that $U = U(G)$ is a REPRESENTATION of the group G if there exists an homomorphism from G to $U(G)$. The dimension of the representation is the dimension of the vector space V.

A representation is FAITHFUL if the homomorphism is an isomorphism ($g_1 \neq g_2 \Rightarrow U(g_1) \neq U(g_2)$). If the representation is not faithful it is said to be DEGENERATE

Suppose that the representation is finite dimensional $\Rightarrow$ we can choose a basis for the vector space $\{e_i\}$, $i = 1 \dots n$

The operator $U(g)$ is then realised as a MATRIX $n \times n$

$$U(g)\,|e_i\rangle = |e_j\rangle\, D(g)^j{}_i$$

EXAMPLE

For every group G there is a trivial representation. Let $V = \mathbb{C}$ and $U(g) = 1$

Clearly $U(g_1)\,U(g_2) = 1 \cdot 1 = 1 = U(g_1 g_2)$ $\Rightarrow$ this is a representation!

(of course not faithful)

EXAMPLE

G group of rotations about a fixed axis

$$R_\theta \cdot R_\varphi = R_{\theta+\varphi} \qquad R_\varphi^{-1} = R_{-\varphi} \qquad R_0 = e \qquad 0 \leq \varphi < 2\pi$$

two-dimensional representation

$$U(R_\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

faithful!

EXAMPLE

$G = D_2 = \{e, h, v, r\}$

h reflection about the vertical axis

v " " " horizontal axis

r rotation by π

$V = \mathbb{R}^2$

$$D(e) = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \qquad D(h) = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \qquad D(v) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \qquad D(r) = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$$

$\Rightarrow$ FAITHFUL REPRESENTATION!

QUOTIENT GROUP AND REPRESENTATIONS

DEF $\quad H \subset G$ subgroup of $G \quad p \in G \quad p \notin H \quad H = \{h_1, h_2 \dots\}$

$pH = \{ph_1, ph_2 \dots\}$ is called LEFT COSET

analogously $\quad Hp = \{h_1 p, h_2 p, \dots\}$ is called RIGHT COSET

LEMMA

Two left cosets of a subgroup H either coincide or don't have common elements. Indeed suppose $ph_i = qh_j$ for some $h_i, h_j \in H$

$\Rightarrow \quad q^{-1}p = h_j h_i^{-1} \in H \quad \Rightarrow \quad q^{-1}p\, H = H \quad \Rightarrow \quad pH = qH \quad \blacksquare$

When H is an invariant subgroup the cosets become particularly useful. First of all left coset are also right coset. Indeed since $pHp^{-1} = H$ (because the subgroup is invariant) we have $pH = Hp$. We can define a new group structure whose elements are the cosets pH. The product is defined as $(pH)\cdot(q\cdot H) \equiv (pq)H$ and $H = eH$ plays the role of neutral element. We can summarize what we have seen with the following

THEOREM $\quad$ If $H \subset G$ is an invariant subgroup the set of cosets of H endowed with the multiplication rule $pH \cdot qH = (pq)H$ forms a group called FACTOR (OR QUOTIENT) GROUP G/H.

The order of G/H is M_G / M_H.

EXAMPLE

Consider the cyclic group C_4. The subgroup $H = \{e, b\}$ is invariant (indeed $aba^{-1} = b$ and $cbc^{-1} = b$)

e	a	b	c
a	b	c	e
b	c	e	a
c	e	a	b

We have $cH = \{c, a\} = aH$.

H and $M = aH$ form the factor group G/H which is isomorphic to C_2

$((aH)\cdot(aH) = a^2H = H)$ $\qquad G/H = \{H, aH\}$ $\qquad \frac{m_G}{m_H} = \frac{4}{2} = 2$ ✓

THEOREM (without proof)

If G has a non-trivial invariant subgroup $H \Rightarrow$ any representation of G/H is also a representation of G, and it is not faithful. Conversely, if $U(G)$ is a non faithful representation of $G \Rightarrow G$ has a non trivial invariant subgroup H such that $U(G)$ provides a faithful representation of G/H.

COROLLARY

All representations (except the trivial one) of SIMPLE GROUPS must be FAITHFUL

EXAMPLE

$G = S_3 \qquad H = \{e, (123), (321)\}$ is invariant $\qquad G/H \sim C_2 = \{e, a\}$

The group C_2 has a representation $\{e, a\} \to \{1, -1\}$. This induces a one dimensional representation of S_3 which assigns 1 to the elements of $H = \{e, (123), (321)\}$ and -1 to $\{(12), (23), (31)\}$

$\Rightarrow$ this method is very useful to construct group representations

EQUIVALENT REPRESENTATIONS

For most of the groups of interest in physics the possible representations can be classified once and for all. To this purpose it is important to distinguish between essentially different (INEQUIVALENT) representations and EQUIVALENT representations.

DEF Let $U(G)$ be a representation of the group G on the vector space V and S an invertible operator on V. Then $U'(G) = S\,U(G)\,S^{-1}$ also forms a representation of G. The relation of the two representations is analogous to the one between two matrices under a change of basis. The two representations $U(G)$ and $U'(G)$ are called EQUIVALENT.

EQUIVALENT representations form equivalence classes.

CHARACTER

The character $\chi(g)$ of $g \in G$ in a representation $U(G)$ is defined as $\chi(g) = \mathrm{Tr}\,(U(g))$. All group elements in a given class have the same character. Indeed $\mathrm{Tr}\left[U(p)\,U(g)\,U(p^{-1})\right] = \mathrm{Tr}\,(U(g))$.

INVARIANT SUBSPACE

Let $U(G)$ be a representation of G on a vector space V. Let $V_1 \subset V$ a subspace of V with the property that $\forall \underline{x} \in V_1$ and $\forall g \in G$ we have $U(g)\,|\underline{x}\rangle \in V_1$

$\Rightarrow$ V_1 is called INVARIANT SUBSPACE

IRREDUCIBLE REPRESENTATIONS

A representation $U(G)$ on a vector space V is said to be IRREDUCIBLE if there exists no invariant subspace in V with respect to $U(G)$. Otherwise the representation is said to be REDUCIBLE

EXAMPLE

$$G = R_\varphi \qquad U_2(G) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

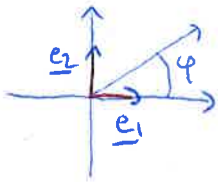
NOTE: $U_{2S} = S U_2 S^{-1}$

S non singular 2×2 matrix

$\rightarrow$ EQUIVALENT TO U_2

This representation is reducible (indeed we are dealing with an abelian group and we will see that all irreducible representations must be one-dimensional)

$$\text{Define} \quad |e_\pm\rangle = \frac{1}{\sqrt{2}}\left(|e_1\rangle \pm i\,|e_2\rangle\right)$$



$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |e_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

do not identify invariant subspaces

But

$$R_\varphi |e_+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^{i\varphi} |e_+\rangle$$

$$R_\varphi |e_-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = e^{-i\varphi} |e_-\rangle$$

$\Rightarrow$ $|e_+\rangle$ and $|e_-\rangle$ identify two invariant subspaces !

Note that

$$\underbrace{U_3(G)}_{\text{3-dimensional}} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is also reducible $\qquad U_3 = U_2 \oplus \underbrace{U_T}_{\text{trivial 1-dim}}$

EXAMPLE

$G = D_2 = \{e, h, v, r\}$ we have seen that

$$U(e) = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \quad U(h) = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \quad U(v) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \quad U(r) = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$$

is a 2-dimensional representation.

$\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\underline{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ span two invariant subspaces

$\Rightarrow$ the representation is REDUCIBLE.

FULLY REDUCIBLE REPRESENTATIONS

A reducible representation is a representation for which there exists an invariant subspace V_1. If the orthogonal complement of V_1 is also invariant the representation is said to be FULLY REDUCIBLE

EXAMPLE

$$U(g) = \left(\begin{array}{c|c} X & X \\ \hline 0 & X \end{array}\right)$$

$X \to$ non vanishing block

invariant

$$U(g) \begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} X \\ 0 \end{pmatrix}$$

reducible!

$$U(g) \begin{pmatrix} 0 \\ X \end{pmatrix} = \begin{pmatrix} X \\ X \end{pmatrix}$$

not invariant

$$U(g) = \left(\begin{array}{c|c} X & 0 \\ \hline 0 & X \end{array}\right)$$

block diagonal form

$$U(g) \begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} X \\ 0 \end{pmatrix}$$

$$U(g) \begin{pmatrix} 0 \\ X \end{pmatrix} = \begin{pmatrix} 0 \\ X \end{pmatrix}$$

fully reducible!

$\Rightarrow$ In the previous example the 2-dim representation of D_2 is FULLY REDUCIBLE

UNITARY REPRESENTATIONS

Suppose that $U(G)$ is a representation on a vector space V with inner product. If the inner product is invariant under G then the representation is said to be UNITARY. Unitary transformations are essential for symmetry groups!

THEOREM

If a unitary representation is REDUCIBLE, then it is FULLY REDUCIBLE

proof

Let $U(G)$ be a reducible representation on the inner-product space V and call V_1 the invariant subspace of dimension m_1. Let us choose $\{\underline{e}_i\}$, $i=1 \cdots m_1$ as a basis of V_1. The orthogonal complement is spanned by $\{\underline{e}_j\}$ where $j = m_1+1 \cdots m$, and is denoted by V_2.

Since V_1 is invariant we have $U(g)|\underline{e}_i\rangle \in V_1$ $\forall g \in G$ with $i=1 \ldots m_1$

But if we denote $|\underline{e}_i(g)\rangle = U(g)|\underline{e}_i\rangle$ we have $\langle \underline{e}_j(g)|\underline{e}_i(g)\rangle =$

$= \langle \underline{e}_j|\underline{e}_i\rangle = 0$ for $i = 1, 2 \cdots m_1$ and $j = m_1+1 \cdots$ because the representation is unitary $\Rightarrow$ $e_j(g)$ is in V_2 $\Rightarrow$ V_2 is also an invariant subspace and $V = V_1 \oplus V_2$ ∎

THEOREM

Every representation of a finite group on a space with inner product is EQUIVALENT TO A UNITARY REPRESENTATION proof

Let $\langle\, , \rangle$ be a inner product on V. We define

$$(v,w) \equiv \langle Sv, Sw\rangle = \sum_{j=1}^{m} \langle U(g_j)\, v\,,\; U(g_j)\, w\rangle$$

m order of the group

One can check that $(\, ,)$ is also a scalar product. The operator S represents the transformation from a basis orthonormal with respect to the scalar product $\langle\, , \rangle$ to another basis orthonormal with respect to the new scalar product $(\, ,)$. We have

$$\big(U(g)\,v,\; U(g)\,w\big) = \sum_{j=1}^{m} \langle U(g)\,U(g_j)\,v\,,\; U(g)\,U(g_j)\,v\rangle$$

$$= \sum_{j=1}^{m} \langle U(g'_j)\,v\,,\; U(g'_j)\,v\rangle = (v,w)$$

rearrangement lemma

$\Rightarrow$ the representation $U(G)$ is then unitary with respect to the new scalar product, which means that it is EQUIVALENT to a unitary representation ■

COMMENT : For this proof we have used the rearrangement lemma, and the fact that the group is finite. The proof, however, suggests that it can be valid also for groups with an infinite number of elements, provided that the summation over the group elements can be properly defined. Examples are the continuous groups of rotations, $U(n)$ and $SU(n)$.

SCHUR'S LEMMA

Let $U_1(G)$ and $U_2(G)$ be two irreducible representations of G on the vector spaces V_1 and V_2. Suppose that T is a linear map from V_1 to V_2 such that

$$U_2(g)\,T = T\,U_1(g) \qquad \forall g \in G$$

$\Rightarrow$ (i) If $U_1 \not\sim U_2$ $\quad T=0$

(ii) If $U_1 = U_2$ (and $V_1 = V_2$) $\Rightarrow$ $T = cI$ for some scalar c

Proof

Let us denote by $\ker T$ the nucleus of T, i.e. the set of vectors $\underline{v} \in V_1 \mid T\underline{v} = 0$. $\ker T$ is an invariant subspace. Indeed if $\underline{v} \in \ker T$ we have $T\,U_1(g)\,\underline{v} = U_2(g)\,T\underline{v} = 0$ and thus $U_1(g)\,\underline{v} \in \ker T$.

Since the representation is irreducible, we must have either $\ker T = V_1$ or $\ker T = \{0\}$. If $\ker T = V_1$ all vectors of V_1 are such that $T\underline{v} = 0$, so we have $T=0$. If $\ker T = \{0\}$ we have that T is an isomorphism, and thus $U_1 \sim U_2$. This establishes (i).

Suppose now that $U_1 = U_2$ (and $V_1 = V_2$). According to the fundamental theorem of algebra, if T is a non trivial linear mapping it will have at least one complex eigenvalue, let us call it c. We then consider $T - cI$ and apply (i) on it. Since $T - cI$ is singular, it cannot be an isomorphism, so we must have $T - cI = 0$ ∎

COROLLARY

Irreducible representations of any abelian group must be of dimension one.

Proof

Let $U(G)$ be an irreducible representation of the abelian group G, and let $p \in G$. We have

$$U(p)\,U(g) = U(pg) = U(gp) = U(g)\,U(p) \qquad \forall g \in G$$

U is a representation

the group is abelian

$\Rightarrow$ according to the Schur's lemma $U(p) = \lambda_p I$. Since this holds $\forall p \in G$ the representation is equivalent to the one-dimensional representation $p \to \lambda_p$ ■

REPRESENTATION THEORY FOR FINITE GROUPS

We now want to present and discuss the basic results for group representation theory. We will use the following notation:

n_G order of the group

μ, ν label inequivalent irreducible representations

n_μ dimension of the μ representation

$D^\mu(g)$ matrix corresponding to g in the representation μ with respect to an orthonormal basis

THEOREM : ORTHONORMALITY OF IRREDUCIBLE REPRESENTATION MATRICES

$$\boxed{\frac{m_\mu}{m_G} \sum_g \left(D^+_\mu(g) \right)_{ki} \left(D^\nu(g) \right)_{je} = \delta_{\mu\nu} \, \delta_{ij} \, \delta_{ke}}$$

Proof

Let X be an arbitrary $m_\mu \times m_\nu$ matrix and define $M_X \equiv \sum_g \underbrace{D^+_\mu(g)}_{m_\mu \times m_\mu} X \underbrace{D_\nu(g)}_{m_\nu \times m_\nu}$

Since we can take the representations unitary we have

$$D^{-1}_\mu(p) \, M_X \, D_\nu(p) = \sum_g D^{-1}_\mu(p) \, D^{-1}_\mu(g) \, X \, D_\nu(g) \, D_\nu(p) = M_X \quad \forall p \in G$$

where we have used the rearrangement lemma.

For the Schur's lemma if $\mu \neq \nu \Rightarrow M_X = 0$ and if $\mu = \nu \Rightarrow M_X = c_X \, I$

Let us choose a family of matrices X_{ke} of the form $(X_{ke})_{ij} = \delta_{kj} \delta_{ie}$

We get

$$\sum_g \left(D^+_\mu(g) \right)_{mi} \left(X_{ke} \right)_{ij} \left(D_\nu(g) \right)_{jn} = \sum_g \left(D^+_\mu(g) \right)_{me} \left(D_\nu(g) \right)_{kn}$$

This must vanish if $\mu \neq \nu$ and must be equal to $c_{ke} \, \delta_{mn}$ if $\mu = \nu$.

The constants c_{ke} can be fixed by taking the trace

$$\underbrace{\sum_g \left(D_\mu(g) D^+_\mu(g) \right)_{ke}}_{= \, m_G \, \delta_{ke}} = \underbrace{m_\mu \, c_{ke}}_{\text{from the trace of } \delta_{mn}} \Rightarrow c_{ke} = \frac{m_G}{m_\mu} \delta_{ke}$$

$$\Rightarrow \sum_g \left(D^+_\mu(g) \right)_{me} \left(D_\nu(g) \right)_{kn} = \delta_{\mu\nu} \frac{m_G}{m_\mu} \delta_{ke} \, \delta_{mn} \qquad \blacksquare$$

This theorem can be regarded as an orthonormality condition of vectors with n_G components. In the case of abelian groups, since irreducible representations are one dimensional, the theorem reduces to

$$\sum_g d_\mu^+(g)\, d_\nu(g) = n_G\, \delta_{\mu\nu} \qquad \text{where } d_\mu(g) \text{ are c-numbers}$$

We can use this theorem to construct new representations from known reps.

EXAMPLE

$G = C_2 = \{e, a\}$ the trivial representation is $(e, a) \xrightarrow{d_1} (1, 1)$

A second representation can be found by requiring the vector to be orthonormal to $(1,1)$ $\Rightarrow$ $(e, a) \xrightarrow{d_2} (1, -1)$. There are no other irreducible representations for C_2. We write

μ	e	a
1	1	1
2	1	-1

EXAMPLE

$G = D_2$ the trivial rep. is $(e, a, b, c) \xrightarrow{d_1} (1, 1, 1, 1)$

$H = \{e, a\}$ is an invariant subgroup.

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

The factor group is $G/H = \{\overset{H}{(e,a)}, \overset{bH}{(b,c)}\} \sim C_2$

$\Rightarrow$ we can use the representations of C_2 to obtain those of D_2

From $(1,1)$ we get the trivial representation d_1. From $(1,-1)$ we get

$(e, a, b, c) \xrightarrow{d_2} (1, 1, -1, -1)$. But also $\{e, b\}$ and $\{e, c\}$ are invariant $\Rightarrow$ we can repeat the construction and we get

$(e, a, b, c) \xrightarrow{d_3} (1, -1, 1, -1)$

$(e, a, b, c) \xrightarrow{d_4} (1, -1, -1, 1)$

$\Rightarrow$ All vectors are orthogonal and there cannot be further representations!

μ	e	a	b	c
1	1	1	1	1
2	1	1	-1	-1
3	1	-1	1	-1
4	1	-1	-1	1

THEOREM : COMPLETENESS OF IRREDUCIBLE REPRESENTATION MATRICES (without proof)

(i) The dimensionality parameters m_μ satisfy $\boxed{\sum_\mu m_\mu^2 = m_G}$

(ii) The representation matrices satisfy the completeness relation

$$\boxed{\sum_{\mu,\ell,k} \frac{m_\mu}{m_G} D^\mu(g)_{\ell k} D^+_\mu(g')_{k\ell} = \delta_{gg'}}$$

For abelian groups we have $m_\mu = 1$ $\forall \mu$ $\Rightarrow$ there must be m_G inequivalent irreducible representations (we have already seen it for C_2 and D_2 !)

The two theorems on the orthonormality and completeness of irreducible representation matrices are important. However the matrices we are working with depend on the choice of a basis. It is then more useful to work with group characters, which do not depend on this choice.

To state the orthonormality and completeness of group characters we need the following lemma.

LEMMA

Let $U^\mu(G)$ an irreducible representation of G and ξ_i a class

$$\Rightarrow \sum_{h\in\xi_i} U^\mu(h) = \frac{m_i}{m_\mu} \chi_i^\mu I$$

m_i number of elements in ξ_i

χ_i^μ character in the class

I identity operator

Proof

Define $A_i \equiv \sum_{h\in\Sigma_i} U^\mu(h) \Rightarrow U^\mu(g) A_i U^\mu(g)^{-1} = A_i \quad \forall g \in G$

since $ghg^{-1} \in \Sigma_i$

$\Rightarrow$ according to the Schur's lemma $A_i = c_i I$. The coefficient c_i can be evaluated by taking the trace $c_i m_\mu = \chi_i^\mu m_i$ ∎

THEOREM : ORTHONORMALITY AND COMPLETENESS OF GROUP CHARACTERS

The characters of inequivalent irreducible representations satisfy the following relations

$$\boxed{\sum_i \frac{m_i}{m_G} \chi_\mu^{+i} \chi_i^\nu = \delta_{\mu\nu} \qquad \text{orthonormality}}$$

sum over classes

$$\boxed{\frac{m_i}{m_G} \sum_\mu \chi_i^\mu \chi_\mu^{+j} = \delta_{ij} \qquad \text{completeness}}$$

sum over inequivalent irreducible represent.

Proof

We start from the orthonormality theorem of irreducible representations

$$\frac{m_\mu}{m_G} \sum_g \left(D_\mu^+(g)\right)_{ki} \left(D_\nu(g)\right)_{j\ell} = \delta_{\mu\nu}\, \delta_{ij}\, \delta_{\ell k}$$

setting $i=k$ and $j=\ell$ the orthonormality of the characters follows by exploiting the fact that the character is the same for all the elements of the same class.

To prove completeness, we start from the completeness relation of the representation matrices:

$$\sum_{\mu,e,k} \frac{m_\mu}{m_G} \left(D^\mu(g)\right)_{ek} \left(D^+_\mu(g')\right)_{ke} = \delta_{gg'}$$

and we sum g over the class i and g' over the class j. Using the previous lemma we obtain

$$\sum_{\mu,e,k} \cancel{\frac{m_\mu}{m_G}} \frac{m_i}{\cancel{m_\mu}} \chi_i^\mu \delta_{ek} \frac{m_j}{m_\mu} \chi_j^{+\mu} \delta_{ke} = m_i \delta_{ij}$$

and thus

$$\frac{m_i}{m_G} \sum_\mu \chi_i^\mu \chi_\mu^{+j} = \delta_{ij} \qquad \blacksquare$$

We now observe that if we define new normalized characters as

$$\tilde{\chi}_i \equiv \sqrt{\frac{m_i}{m_G}}\, \chi_i$$

we can write the previous orthonormality and completeness relations as

$$\tilde{\chi}_\mu^{+i} \tilde{\chi}_i^\nu = \delta_{\mu\nu}$$

$$\tilde{\chi}_i^\mu \tilde{\chi}_\mu^{+j} = \delta_{ij}$$

Thus $\tilde{\chi}_i^\mu$ can be interpreted as a vector with $i = 1, 2 \cdots M_C$ component for the irreducible rep. μ, or as a vector with $\mu = 1, 2 \cdots N$ components for the class i (N being the number of irreducible representations)

Now since the number of mutually orthogonal vectors cannot be larger than the dimension of the vector space, the first relation implies that $N \leq M_C$.

On the other hand the second relation implies that $M_C \leq N$

$\Rightarrow$ The number of irreducible representations of any finite group G is equal to the number of distinct classes M_C

As a consequence of this important result, the characters can be grouped in a CHARACTER TABLE

$\mu \backslash i$	1	2	3	...
1				
2				
3				
⋮				

← class index

↑ representation index

In the case of abelian groups, since each element is itself an independent class, we recover the previous result that the number of irreducible representations is equal to the number of group elements M_G

Characters are simpler than the representation matrices and much more useful!

EXAMPLE: IRREDUCIBLE REPRESENTATIONS OF S_3

Let us find the irreducible representations of S_3. This group has three classes:

1-cycle $\{e\}$ $\quad$ 2-cicle $\{(12), (23), (31)\}$ $\quad$ 3-cycle $\{(123), (321)\}$

$i=1$ $\quad$ $i=2$ $\quad$ $i=3$

$\Rightarrow$ we expect three irreducible representations.

Let us label with $\mu=1$ the trivial representation. The three characters are $(1,1,1)$. Since S_3 has the invariant subgroup $H=\{e, (123), (321)\}$. We have seen that the quotient group is $S_3/H \sim C_2$. The non-trivial C_2 representation $\{e,e\} \to \{1,-1\}$ induces a representation of S_3 which is one dimensional, and which maps the elements of H into 1 and the remaining elements (the 2-cicles (12), (23) and (31)) into -1. This representation has characters $(1,-1,1)$ for 1, 2 and 3-cycles: we label it with $\mu=2$. Since $m_G=6$ and there are 3 irreducible representations, the last one must be 2 dimensional $\left(\sum_\mu m_\mu^2 = m_G = 6\right)$. Since it is 2 dimensional, the character corresponding to the identity must be equal to 2. The remaining characters can be deduced by the orthogonality conditions

$\mu \backslash i$	1	2	3
1	1	1	1
2	1	-1	1
3	2	α	β

character table

- Orthogonality $\mu=3$, $\mu=1$

$$\frac{1}{6}2\cdot 1 + \frac{3}{6}\cdot\alpha + \frac{2}{6}\cdot\beta = 0$$

- Orthogonality $\mu=3$ $\mu=2$

$$\frac{1}{6}2\cdot 1 + \frac{(-3)\alpha}{6} + \frac{2}{6}\beta = 0$$

$\Rightarrow$ $\alpha = 0$, $\beta = -1$

EXAMPLE : IRREDUCIBLE REPRESENTATIONS OF C_3

C_3 is abelian : 3 classes and 3 1-dimensional irreducible representations

$\Rightarrow$ the structure of the character table is

	e	a	b
1	1	1	1
2	1	α	β
3	1	γ	δ

$\alpha, \beta, \gamma, \delta \in \mathbb{C}$ to be determined

From the orthonormality of the characters we find the following constraints

$$\tilde{\chi}_2^+ \cdot \tilde{\chi}_2 = \frac{1}{3}\left(1 + |\alpha|^2 + |\beta|^2\right) = 1$$

$$\tilde{\chi}_3^+ \cdot \tilde{\chi}_3 = \frac{1}{3}\left(1 + |\gamma|^2 + |\delta|^2\right) = 1$$

$$\tilde{\chi}_2^+ \cdot \tilde{\chi}_1 = \frac{1}{3}\left(\alpha^* + \beta^* + 1\right) = 0$$

$$\tilde{\chi}_3^+ \cdot \tilde{\chi}_1 = \frac{1}{3}\left(\gamma^* + \delta^* + 1\right) = 0$$

$$\tilde{\chi}_3^+ \cdot \tilde{\chi}_2 = \frac{1}{3}\left(\gamma^*\alpha + \delta^*\beta + 1\right) = 0$$

$$\Rightarrow \left\{ \begin{array}{l} |\alpha|^2 + |\beta|^2 = 2 \\ |\gamma|^2 + |\delta|^2 = 2 \\ \alpha^* + \beta^* = -1 \\ \gamma^* + \delta^* = -1 \\ \alpha\gamma^* + \beta\delta^* = -1 \end{array} \right.$$

$$\Rightarrow \quad \alpha = e^{2\pi i/3} \qquad \beta = \alpha^* = e^{-2\pi i/3}$$

$$\gamma = \alpha^* \qquad \delta = \alpha$$

THEOREM

In the reduction of a given representation of a group G into irreducible representations, the number of times an irreducible representation appears is given by

$$a_\mu = \sum_i \frac{m_i}{m_G} \chi_\mu^{i+} \chi_i = \tilde{\chi}_\mu^+ \cdot \tilde{\chi}$$

Proof

We have $U(G) = \sum_\mu a_\mu U^\mu(G)$ and $\tilde{\chi}_i = \sum_\mu a_\mu \tilde{\chi}_i^\mu$.

If we multiply by $\tilde{\chi}_\nu^+$ we get

$$\tilde{\chi} \cdot \tilde{\chi}_\nu^+ = \sum_\mu a_\mu \tilde{\chi}^\mu \tilde{\chi}_\nu^+ = a_\nu$$

where we have exploited the orthonormality of the characters ■

EXAMPLE

$G = C_2$ $\quad D(e) = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ $\quad D(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

It is a 2-dimensional representation $\Rightarrow$ REDUCIBLE!

$\chi = (2,0)$ $\quad \tilde{\chi} = (\sqrt{2}, 0)$ $\quad \tilde{\chi}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\tilde{\chi}_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

	e	a
1	1	1
2	1	-1

$$a_1 = \tilde{\chi} \cdot \tilde{\chi}_1 = 1$$

$$a_2 = \tilde{\chi} \cdot \tilde{\chi}_2 = 1$$